Bethe ansatz method in Gaudin Model

For the cases of classical simple Lie algebras

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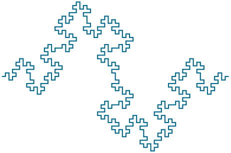

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SIMPLE LIE ALGEBRA $\mathfrak{g}$

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank r. Denote the universal enveloping algebra of $\mathfrak{g}$ by $\mathcal{U}(\mathfrak{g})$. Let $(\,,\,)$ be the Killing form of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ be the Cartan decomposition. Fix simple roots $\alpha_1, \dots, \alpha_r$ in $\mathfrak{h}^*$. Let $\alpha_1^\vee, \dots, \alpha_r^\vee \in \mathfrak{h}$ be the corresponding coroots. Let $\omega_1, \dots, \omega_r \in \mathfrak{h}^*$ be the fundamental weights, $\langle \omega_j, \alpha_i^\vee \rangle = \delta_{i,j}$.

Let $\rho \in \mathfrak{h}^*$ be such that $\langle \rho, \alpha_i^\vee \rangle = 1$, $i = 1, \dots, r$.

For $\lambda \in \mathfrak{h}^*$, let V_λ be the irreducible $\mathfrak{g}$-module with highest weight λ.

SIMPLE LIE ALGEBRA $\mathfrak{g}$

Let V be a $\mathfrak{g}$-module. Let $V^{\mathrm{sing}} = \{v \in V \mid \mathfrak{n}_+ v = 0\}$ be the **singular space** in V. For $\mu \in \mathfrak{h}^*$ let $(V)_\mu = \{v \in V \mid hv = \langle \mu, h \rangle v\}$ be the subspace of V of vectors of **weight** μ. Let $(V)_\mu^{\mathrm{sing}} = (V)_\mu \cap V^{\mathrm{sing}}$ be the **singular space of weight** μ in V.

The coproduct $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is defined to be the homomorphism of algebras such that $\Delta x = 1 \otimes x + x \otimes 1$, for all $x \in \mathfrak{g}$.

Let $(x_i)_{i \in O}$ be an orthonormal basis of $\mathfrak{g}$ w.r.t the Killing-form .
Let $\Omega = \sum_{i \in O} x_i \otimes x_i \in \mathfrak{g} \otimes \mathfrak{g} \subset \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$.

For any $u \in \mathcal{U}(\mathfrak{g})$, we have $\Delta(u)\Omega = \Omega\Delta(u)$.

TENSOR PRODUCT OF VECTOR REPRESENTATIONS

Let n be a positive integer and $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$, $\Lambda_i \in \mathfrak{h}^*$, a sequence of weights. Denote by $V_{\boldsymbol{\Lambda}}$ the $\mathfrak{g}$-module $V_{\Lambda_1} \otimes \cdots \otimes V_{\Lambda_n}$.

If $X \in \mathrm{End}(V_{\Lambda_i})$, then we denote by $X^{(i)} \in \mathrm{End}(V_{\boldsymbol{\Lambda}})$ the operator

$$\mathrm{id}^{\otimes i-1} \otimes X \otimes \mathrm{id}^{\otimes n-i}$$

acting non-trivially on the i-th factor of the tensor product.

If $X = \sum_k X_k \otimes Y_k \in \mathrm{End}(V_{\Lambda_i} \otimes V_{\Lambda_j})$, then we set

$$X^{(i,j)} = \sum_k X_k^{(i)} \otimes Y_k^{(j)} \in \mathrm{End}(V_{\boldsymbol{\Lambda}}).$$

GAUDIN MODEL

Let $\boldsymbol{z} = (z_1, \dots, z_n)$ be a point in $\mathbb{C}^n$ with distinct coordinates. Introduce linear operators $\mathcal{H}_1(\boldsymbol{z}), \dots, \mathcal{H}_n(\boldsymbol{z})$ on $V_{\boldsymbol{\Lambda}}$ by the formula

$$\mathcal{H}_i(\boldsymbol{z}) = \sum_{j,\ j \neq i} \frac{\Omega^{(i,j)}}{z_i - z_j}, \quad i = 1, \dots, n.$$

The operators $\mathcal{H}_1(\boldsymbol{z}), \dots, \mathcal{H}_n(\boldsymbol{z})$ are called the **Gaudin Hamiltonians** of the Gaudin model associated with $V_{\boldsymbol{\Lambda}}$.

Proposition

The Gaudin Hamiltonians commute, $[\mathcal{H}_i(\boldsymbol{z}), \mathcal{H}_j(\boldsymbol{z})] = 0$ for all i, j. The Gaudin Hamiltonians commute with the action of $\mathfrak{g}$, $[\mathcal{H}_i(\boldsymbol{z}), x] = 0$ for all i and $x \in \mathfrak{g}$. In particular, for any $\mu \in \mathfrak{h}^*$, the Gaudin Hamiltonians preserve the subspace $(V_{\boldsymbol{\Lambda}})^{\mathrm{sing}}_{\mu} \subset V_{\boldsymbol{\Lambda}}$.

Gaudin model

One of the main problems in the Gaudin model is to find common eigenvectors and eigenvalues of the Gaudin Hamiltonians. By the proposition, it suffices to do that in the subspace $(V_{\boldsymbol{\Lambda}})^{\mathrm{sing}}_{\Lambda_\infty}$.

The main method is the **algebraic Bethe ansatz method**.

The idea of this method is to find a vector-valued function of a special form and determine its arguments in such a way that the value of this function is an eigenvector.

The function is called **weight function**. The equations which determine the special values of arguments are called the **Bethe ansatz equations**.

GENERAL MASTER FUNCTION

Fix a simple Lie algebra $\mathfrak{g}$, a sequence of weights $\mathbf{\Lambda} = (\Lambda_i)_{i=1}^n$, $\Lambda_i \in \mathfrak{h}^*$, and a sequence of non-negative integers $\boldsymbol{l} = (l_1, \ldots, l_r)$.

Denote $l = l_1 + \cdots + l_r$, $\alpha(\boldsymbol{l}) = l_1\alpha_1 + \cdots + l_r\alpha_r$, and $\Lambda_\infty = \Lambda_1 + \cdots + \Lambda_n - \alpha(\boldsymbol{l})$. (We will consider $(V_{\mathbf{\Lambda}})^{\text{sing}}_{\Lambda_\infty}$.)

Introduce the **master function** $\Phi_{\mathfrak{g},\mathbf{\Lambda},\boldsymbol{l}}(\boldsymbol{t};\boldsymbol{z})$ which is a function of l variables

$$\boldsymbol{t} = (t_1^{(1)}, \ldots, t_{l_1}^{(1)}; \ldots; t_1^{(r)}, \ldots, t_{l_r}^{(r)})$$

by the formula

$$\Phi_{\mathfrak{g},\mathbf{\Lambda},\boldsymbol{l}}(\boldsymbol{t};\boldsymbol{z}) = \prod_{1\leqslant i<j\leqslant n} (z_i - z_j)^{(\Lambda_i,\Lambda_j)} \prod_{i=1}^{r} \prod_{j=1}^{l_i} \prod_{s=1}^{n} (t_j^{(i)} - z_s)^{-(\Lambda_s,\alpha_i)}$$
$$\times \prod_{i=1}^{r} \prod_{1\leqslant j<s\leqslant l_i} (t_j^{(i)} - t_s^{(i)})^{(\alpha_i,\alpha_i)} \prod_{1\leqslant i<j\leqslant r} \prod_{s=1}^{l_i} \prod_{k=1}^{l_j} (t_s^{(i)} - t_k^{(j)})^{(\alpha_i,\alpha_j)}.$$

BETHE ANSATZ EQUATION

A point $\boldsymbol{t} \in \mathbb{C}^l$ is called a **critical point** of $\Phi(\cdot\,; \boldsymbol{z})$, if

$$\left(\Phi^{-1}\frac{\partial \Phi}{\partial t_j^{(i)}}\right)(\boldsymbol{t};\boldsymbol{z}) = 0, \quad \text{for } i = 1,\ldots,r \text{ and } j = 1,\ldots,l_i.$$

In other words, the following system of algebraic equations is satisfied,

$$0 = \sum_{s=1}^{n}\frac{-(\Lambda_s,\alpha_i)}{t_j^{(i)} - z_s} + \sum_{s=1,s\neq j}^{l_i}\frac{(\alpha_i,\alpha_i)}{t_j^{(i)} - t_s^{(i)}} + \sum_{k=1,k\neq i}^{r}\sum_{s=1}^{l_k}\frac{(\alpha_i,\alpha_s)}{t_j^{(i)} - t_s^{(k)}}.$$

This system of the equations is called the **Bethe ansatz equation** associated to $\mathfrak{g}$-module $V_{\boldsymbol{\Lambda}}$.

The product of symmetric group $\boldsymbol{S_l} = S_{l_1} \times \cdots \times S_{l_r}$ acts on the variables $\boldsymbol{t}$ by permuting the coordinates with the same upper index. The master function is $\boldsymbol{S_l}$-invariant. The set of critical points of $\Phi(\cdot\,;\boldsymbol{z})$ is $\boldsymbol{S_l}$-invariant. We will not distinguish critical points in the same $\boldsymbol{S_l}$-orbit.

For a critical point $\boldsymbol{t}$, define the tuple $\boldsymbol{y}^{\boldsymbol{t}} = (y_1, \dots, y_r)$ of polynomials of x by

$$y_i(x) = \prod_{j=1}^{l_i}(x - t_j^{(i)}) \quad \text{for } i = 1, \dots, r.$$

Solving BAE is equivalent to finding coefficients of y_i. We say that $\boldsymbol{y}^{\boldsymbol{t}}$ **represents the critical point** $\boldsymbol{t}$.

WEIGHT FUNCTION

The formula for the Bethe vector is a rational map

$$\omega : \mathbb{C}^l \times \mathbb{C}^n \to (V_{\boldsymbol{\Lambda}})_{\Lambda_\infty}, \quad (\boldsymbol{t}, \boldsymbol{z}) \mapsto \omega(\boldsymbol{t}; \boldsymbol{z})$$

called the **canonical weight function**, which was introduced by [Matsuo 1990] for $\mathfrak{gl}_{r+1}$ and by [Schechtman-Varchenko 1991] for all simple Lie algebras.

Let $\boldsymbol{t} \in \mathbb{C}^l$ be a critical point of the master function $\Phi(\cdot\,;\boldsymbol{z})$. Then the value of the weight function $\omega(\boldsymbol{t}\,;\boldsymbol{z}) \in (V_{\boldsymbol{\Lambda}})_{\Lambda_\infty}$ is called the **Bethe vector**.

Lemma (Mukhin-Varchenko 2004)

If Λ_∞ is dominant integral, then the set of critical points is finite.

Assume that $\boldsymbol{t} \in \mathbb{C}^l$ is an isolated critical point of the master function $\Phi(\cdot\,;\boldsymbol{z})$.

Theorem (Varchenko 2011)

The Bethe vector $\omega(\boldsymbol{t};\boldsymbol{z})$ is non-zero.

Theorem (Reshetikhin-Varchenko 1995)

The Bethe vector $\omega(\boldsymbol{t};\boldsymbol{z})$ is singular, $\omega(\boldsymbol{t};\boldsymbol{z}) \in (V_{\boldsymbol{\Lambda}})^{\mathrm{sing}}_{\Lambda_\infty}$. Moreover, $\omega(\boldsymbol{t};\boldsymbol{z})$ is a common eigenvector of the Gaudin Hamiltonians,

$$\mathcal{H}_i(\boldsymbol{z})\omega(\boldsymbol{t};\boldsymbol{z}) = \left(\Phi^{-1}\frac{\partial\Phi}{\partial z_i}\right)(\boldsymbol{t};\boldsymbol{z})\omega(\boldsymbol{t};\boldsymbol{z}), \qquad i = 1,\ldots,n.$$

Completeness

Bethe ansatz conjecture

If $z_1, \dots, z_n$ are generic and $\Lambda_1, \dots, \Lambda_n$ are dominant integral. Then the number of solutions of BAE equals to $\dim(V_{\boldsymbol{\Lambda}})^{\mathrm{sing}}_{\Lambda_\infty}$ and the Bethe vectors obtained from those solutions form a basis of $(V_{\boldsymbol{\Lambda}})^{\mathrm{sing}}_{\Lambda_\infty}$.

The conjecture is **false**. In [Mukhin-Varchenko 2007], there is a counterexample for which the conjecture is false for all $\boldsymbol{z}$.

When $\boldsymbol{z}$ is generic, the conjecture is true for the following cases with certain tensor products

- Lie algebra $\mathfrak{gl}_{r+1}$ [Mukhin-Varchenko 2005]
- Lie superalgebra $\mathfrak{gl}(m|n)$ [Mukhin-Vicedo-Young 2015]
- Lie algebras of types B,C,D [L-Mukhin-Varchenko]

COMPLETENESS FOR TYPES B,C,D

Theorem (L-Mukhin-Varchenko)

Let $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or $\mathfrak{so}_{2r}$ or $\mathfrak{sp}_{2r}$ and let λ be dominant integral. For generic $\boldsymbol{z}$ there exists a set of solutions $\{\boldsymbol{t}_i,\ i \in I\}$ of the Bethe ansatz equation such that the corresponding Bethe vectors $\{\omega(\boldsymbol{t}_i; \boldsymbol{z}),\ i \in I\}$ form a basis of $\left(V_\lambda \otimes V_{\omega_1}^{\otimes n}\right)^{\mathrm{sing}}$. If $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or $\mathfrak{sp}_{2r}$, the Gaudin Hamiltonians have simple joint spectrum.

The spectrum is not simple for type D since the Dynkin diagram for type D has a nontrivial symmetry.

By the standard methods, the completeness for $V_\lambda \otimes V_{\omega_1}^{\otimes n}$ is reduced to the case of $n = 1$.

The reduction involves taking appropriate limits, when all points z_i go to the same number with different rates.

COMPLETENESS FOR TYPES B,C,D

For 2-point case, we can always rescale (z_1, z_2) to $(0, 1)$. Note that the decomposition of $V_\lambda \otimes V_{\omega_1}$ is **multiplicity-free** for all dominant integral weights λ. We expect to solve the BAE explicitly.

This is equivalent to finding the coefficients of a tuple of polynomials $\boldsymbol{y}^{\boldsymbol{t}}$, which represents this solution. In all previously known results for the multiplicity-free cases, those coefficients can be completely factorized into products of linear functions of the parameters with integer coefficients.

The difficulty for types B,C,D is that the coefficients can not be factorized in such a fashion.

Our idea comes from the reproduction procedure studied in [Mukhin-Varchenko 2008]. This reproduction procedure allows us to reduce the problem to the trivial case $\boldsymbol{l} = (0, \dots, 0)$ with different initial data.

By solving the BAE, we obtain that the constant terms of those polynomials whose degrees are at most 2 are factorizable while the linear coefficients can be expressed as sums of two factorizable terms.

Let $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$. For the case $\boldsymbol{l} = (2, \ldots, 2)$, the constant term of the quadratic polynomial y_k, $k = 1, \ldots, r$, is

$$\left(\prod_{j=1}^{2r-l} \frac{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1}{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j} \right)^2$$

$$\times \prod_{j=2r-l+1}^{r-1} \frac{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 2}{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1}$$

$$\times \prod_{j=2r-l+1}^{k-1} \frac{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 2}{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1}$$

$$\times \prod_{i=1}^{r-k} \frac{\lambda_{2r-l+1} + \cdots + \lambda_{r-i} + l - r - i - 1}{\lambda_{2r-l+1} + \cdots + \lambda_{r-i} + l - r - i}$$

$$\times \frac{2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2l - 2r - 3}{2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2l - 2r - 1}.$$

The linear coefficient of the quadratic polynomial y_k is

$$\frac{2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+2l-2r-3}{2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+2l-2r-2}$$

$$\times\prod_{j=1}^{2r-l}\frac{\lambda_j+\cdots+\lambda_{2r-l}+2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-1}{\lambda_j+\cdots+\lambda_{2r-l}+2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j}$$

$$\times\left(\prod_{j=2r-l+1}^{k-1}\frac{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-2}{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-1}\right.$$

$$+\prod_{j=2r-l+1}^{r-1}\frac{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-2}{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-1}$$

$$\left.\times\prod_{i=1}^{r-k}\frac{\lambda_{2r-l+1}+\cdots+\lambda_{r-i}+l-r-i-1}{\lambda_{2r-l+1}+\cdots+\lambda_{r-i}+l-r-i}\right).$$

Thank you!